Chapter 6

Series and Residues

6.1 Sequences and Series

- A **sequence** $\{z_n\}$ is a function whose domain is the set of positive integers and whose range is a subset of the complex numbers **C**.

(ex.) The sequence $\{1+i^n\}$ is

$$\begin{array}{cccccc} 1+i, & 0, & 1-i, & 2, & 1+i, & \cdots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ n=1, & n=2, & n=3, & n=4, & n=5, & \cdots \end{array}$$

- If $\lim_{n\to\infty} z_n = L$, we say the sequence $\{z_n\}$ is **convergent**.
- A sequence that is not convegent is said to be **divergent**.

[Theorem 6.1] Criterion for convergence

- A sequence $\{z_n\}$ converges to a complex number $L = a + ib$ if and only if $\mathrm{Re}(z_n)$ converges to $\mathrm{Re}(L) = a$ and $\mathrm{Im}(z_n)$ converges to $\mathrm{Im}(L) = b$.

- An **infinite series** or **series** of complex numbers

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + z_3 + \cdots + z_n + \cdots$$

is convergent if the sequence of partial sums $\{S_n\}$, where

$$S_n = z_1 + z_2 + z_3 + \cdots + z_n,$$

converges.

- A **geometric series** is any series of the form

$$\sum_{k=1}^{\infty} az^{k-1} = a + az + az^2 + \cdots + az^{n-1} + \cdots$$

The *n*th term of the sequence of partial sums is

$$S_n = a + az + az^2 + \cdots + az^{n-1}$$

$$zS_n = az + az^2 + az^3 + \cdots + az^n$$

$$\begin{aligned} S_n - zS_n &= \left(a + az + az^2 + \cdots + az^{n-1}\right) \\ &\quad - \left(az + az^2 + az^3 + \cdots + az^n\right) \\ &= a - az^n \end{aligned}$$

$$S_n = \frac{a\left(1 - z^n\right)}{1 - z}$$

(i) $z^n \to 0$ as $n \to \infty$ whenever $|z| < 1$, and so

$$S_n \to \frac{a}{1 - z}$$

(ii) A geometric series $\sum_{k=1}^{\infty} az^{k-1}$ diverges when $|z| \geq 1$.

[Theorem 6.2] A necessary condition for convergence

- If $\sum_{k=1}^{\infty} z_k$ converges, then $\lim_{n \to \infty} z_n = 0$.

- An infinite series of the form

$$\sum_{k=0}^{\infty} a_k \left(z-z_0\right)^k = a_0 + a_1\left(z-z_0\right) + a_2\left(z-z_0\right)^2 + \cdots \tag{11}$$

  where the coefficient a_k are complex constants, is called a **power series** in $z-z_0$. The complex point z_0 is referred to as the **center** of the series.

- A complex power series (11) has a **circle of convergence**, which is the circle centered at z_0 of largest radius $R>0$ for which (11) converges at every point within the circle $\left|z-z_0\right|<R$. (Refer to Figure 6.3)

  (i) $R=0$: converge at $z=z_0$

  (ii) R: a finite positive number at all interior points of the circle $\left|z-z_0\right|=R$

  (ii) $R=\infty$: converge for all z

6.2 Taylor Series

[Theorem 6.6] Continuity

- A power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ represents a continuous function f within its circle of convergence $|z - z_0| = R$.

[Theorem 6.7] Term-by-term differentiation

- A power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ can be differentiated term by term within its circle of convergence $|z - z_0| = R$.

[Theorem 6.8] Term-by-term integration

- A power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ can be integrated term by term within its circle of convergence $|z - z_0| = R$, for every contour C lying entirly within the circle of convergence

- Suppose a power series represents a function f within $|z - z_0| = R$, that is,

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

$$f'(z) = \sum_{k=1}^{\infty} a_k k (z - z_0)^{k-1} = 1 \cdot a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + \cdots$$

$$f''(z) = \sum_{k=2}^{\infty} a_k k(k-1)(z - z_0)^{k-2} = 2 \cdot 1a_2 + 3 \cdot 2 \cdot 1a_3(z - z_0) + \cdots$$

$$f'''(z) = \sum_{k=3}^{\infty} a_k k(k-1)(k-2)(z - z_0)^{k-3} = 3 \cdot 2 \cdot 1a_3 + \cdots$$

A power series represents an analytic function within its circle of convergence.

$$\begin{aligned} f(z_0) &= a_0 \\ f'(z_0) &= 1!a_1 \\ f''(z_0) &= 2!a_2 \\ f'''(z_0) &= 3!a_3 \\ &\vdots \end{aligned}$$

$$f^{(n)}(z_0) = n!a_n \quad => \quad a_n = \frac{f^{(n)}(z_0)}{n!}, \; n \ge 0$$

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k$$

This series is called the **Taylor series** for *f* centered at z_0.

$$f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}z^k$$

is referred to as a **Maclaurin series**.

[Theorem 6.9] Taylor's Theorem

- Let f be analytic within a domain *D* and let z_0 be a point in *D*. Then, f has the series representation

$$f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}\left(z-z_0\right)^k$$

valid for the largest circle *C* with center at z_0 and radius *R* that lies entirely within *D*.

- Note that the radius of convergence *R* is the distance from the center z_0 of the series to the nearest **isolated singularity** of *f*.

- Some important Maclaurin Series

$$e^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\cdots=\sum_{k=0}^{\infty}\frac{z^k}{k!}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$$

6.3 Laurent Series

- If a complex function f fails to be analytic at a point $z = z_0$, then this point is said to be a **singularity** or **singular point** of the function.

- Suppose that $z = z_0$ is a singularity of a complex function f.
  The point $z = z_0$ is said to be an **isolated singularity** of the function f if there exists some deleted neghborhood, or punctured open disk, $0 < |z - z_0| < R$ of z_0 throughout which f is analytic.

- About an isolated singularity $z = z_0$, it is possible to represent f by a series involving both negative and nonnegative integer powers of $z - z_0$;

< Laurent series expansion >

$$f(z) = \cdots + \frac{a_{-2}}{\left(z-z_0\right)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1\left(z-z_0\right) + a_2\left(z-z_0\right)^2 + \cdots$$

$$= \sum_{k=-\infty}^{\infty} a_k \left(z-z_0\right)^k$$

$$f(z) = \underbrace{\sum_{k=1}^{\infty} a_{-k}\left(z-z_0\right)^{-k}}_{\textbf{principal part} \text{ of the series}} + \underbrace{\sum_{k=0}^{\infty} a_k\left(z-z_0\right)^k}_{\textbf{analytic part} \text{ of the series}}$$

[Theorem 6.10] Laurent's Theorem

- Let f be analytic within a domain D defined by $r < \left| z-z_0 \right| < R$. Then f has the series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k\left(z-z_0\right)^k$$

valid for $r < \left| z-z_0 \right| < R$. The coefficients a_k are given by

$$a_k = \frac{1}{2\pi i}\oint_C \frac{f(s)}{(s-z_0)^{k+1}}\,ds, \; k = 0, \pm 1, \pm 2, \cdots$$

where C is a simple closed curve that lies entirely within D and has z_0 in its interior. (See Figure 6.6)

6.4 Zeros and Poles

- A number z_0 is **zero** of a function f if $f(z_0) = 0$.

- An isolated singularity point $z = z_0$ of a complex function f is given a classification depending on whether the principal part of its Laurent expansion contains zero, a finite number, or an infinite number of terms.
  - (i) If the principal part is zero, that is all the coefficients a_{-k} are zero, then $z = z_0$ is called a **removable singularity**.
  - (ii) If the principal part contains a finite number of nonzero terms, then $z = z_0$ is called a **pole**.
    If the last nonzero coefficient is a_{-n}, $n \geq 1$, then $z = z_0$ is a **pole of order *n***.
    A pole of order 1 => a **simple pole**.
  - (iii) If the principal part contains an infinitely many nonzero terms, then $z = z_0$ is called an **essential singularity**.

- A zero of order n is referred to as a **zero of multiplicity *n***.
- An analytic function f has a **zero of order *n*** at $z = z_0$ if
  $f(z_0) = 0$, $f'(z_0) = 0$, $f''(z_0) = 0$, …..
  $f^{(n-1)}(z_0) = 0$, but $f^{(n)}(z_0) \neq 0$.

[Theorem 6.11] Zero of order *n*

- A function f that is analytic in some disk $|z - z_0| < R$ has a zero of order n at $z = z_0$ if and only if f can be written

$$f(z) = (z - z_0)^n f(z)$$

where f is analytic at $z = z_0$ and $f(z_0) \neq 0$.

[Theorem 6.12] Pole of order *n*

- A function f analytic in a punctured disk $0 < |z - z_0| < R$ has a pole of order n at $z = z_0$ if and only if f can be written

$$f(z) = \frac{f(z)}{(z - z_0)^n}$$

where f is analytic at $z = z_0$ and $f(z_0) \neq 0$.

[Theorem 6.13] Pole of order *n*

- If functions g and h are analytic at $z = z_0$ and h has a zero of order *n* at $z = z_0$ and $g(z_0) \neq 0$, then the function

$$f(z) = \frac{g(z)}{h(z)}$$

has a pole of order *n* at $z = z_0$.

6.5 Residues and Residue Theorem

If a complex function f has an isolated singularity at a point z_0, then f has a Laurent series representation.

$$f(z) = \sum_{k=-\infty}^{\infty} a_k \left(z - z_0\right)^k$$

$$= \cdots + \frac{a_{-2}}{\left(z - z_0\right)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1\left(z - z_0\right) + a_2\left(z - z_0\right)^2 + \cdots$$

which is valid for $0 < \left| z - z_0 \right| < R$.

- The coefficient a_{-1} of $\dfrac{1}{z - z_0}$ in the Laurent series is called the **residue** of the function f at the isolated singularity z_0.

  $a_{-1} = \operatorname{Res}\left(f(z),\, z_0 \right)$: Residue of f at z_0

[Theorem 6.14] Residue at a simple pole

- If f has a simple pole at $z = z_0$, then

$$a_{-1} = \text{Res}\left(f(z),\, z_0 \right) = \lim_{z \to z_0} \left(z - z_0 \right) f(z)$$

[Theorem 6.15] Residue at a pole of order *n*

- If f has a pole of order n at $z = z_0$, then

$$a_{-1} = \text{Res}\left(f(z),\, z_0 \right) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left(z - z_0 \right)^n f(z)$$

- An alternative method for computing a residue at a simple pole :

  Suppose a function f can be written as a quotient $f(z) = \dfrac{g(z)}{h(z)}$, where g and h are analytic at $z = z_0$.

  If $g(z_0) \neq 0$ and if the function h has a zero of order 1 at z_0, then f has a simple pole at $z = z_0$ and

$$a_{-1} = \text{Res}\left(f(z),\, z_0 \right) = \frac{g(z_0)}{h'(z_0)}$$

[Theorem 6.16] Cauchy's residue theorem

- Let *D* be a simply connected domain and *C* a simple closed contour lying entirely within *D*. If a function f is analytic on and within *C*, except at finite number of isolated singular points $z_1, z_2, \ldots.. z_n$ within *C*, then

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\left(f(z), z_k \right)$$

(Example 4) Evaluation by the residue theorem

Evaluate $\oint_C \frac{1}{(z-1)^2(z-3)}\,dz$, where

(a) the contour C is the rectangle defined by $x=0,\ \ x=4,\ \ y=-1\ \ y=1,$

(b) and the contour C is the circle $|z|=2$.

<u>*Solution*</u>*:*

(a) Since both $z=1$ (n=2) and $z=3$ (n=1) are poles within the rectangle,

$$\oint_C \frac{1}{(z-1)^2(z-3)}\,dz = 2\pi i\left[\operatorname{Res}\big(f(z),1\big)+\operatorname{Res}\big(f(z),3\big)\right]$$

$$= 2\pi i\left[\frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}(z-1)^2\frac{1}{(z-1)^2(z-3)}+\lim_{z\to 3}(z-3)\frac{1}{(z-1)^2(z-3)}\right.$$

$$= 2\pi i\left[-\frac{1}{4}+\frac{1}{4}\right]=0$$

(b) Since only the pole $z=1$ lies within the circle $|z|=2$, (n=2)

$$\oint_C \frac{1}{(z-1)^2(z-3)}\,dz = 2\pi i\ \operatorname{Res}\big(f(z),1\big)$$

$$= 2\pi i\frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}(z-1)^2\frac{1}{(z-1)^2(z-3)}$$

$$= 2\pi i\left(-\frac{1}{4}\right)=-\frac{\pi}{2}i$$

(Example 5) Evaluation by the residue theorem

Evaluate $\oint_C \frac{2z+6}{z^2+4}dz$, where the contour C is the circle $|z-i|=2$.

<u>Solution</u>:

$$z^2+4=(z-2i)(z+2i)$$

So, the integrand has simple poles at $-2i$ and $2i$. (n=1)

Because only $2i$ lies within the contour C,

$$\oint_C \frac{2z+6}{z^2+4}dz = 2\pi i \operatorname{Res}\big(f(z), 2i\big)$$

$$= 2\pi i \lim_{z\to 2i}(z-2i)\frac{2z+6}{(z-2i)(z+2i)}$$

$$= 2\pi i \frac{6+4i}{4i} = 2\pi i \frac{3+2i}{2i}$$

$$= \pi(3+2i)$$

(Example 6) Evaluation by the residue theorem

Evaluate $\oint_C \dfrac{e^z}{z^4 + 5z^3}\, dz$, where the contour C is the circle $|z| = 2$.

Solution:

$$z^4 + 5z^3 = z^3(z+5)$$

So, the integrand $f(z)$ has a pole of order 3 at $z = 0$ and a simple pole at $z = -5$.

Only the pole $z = 0$ lies within the given contour (n=3) and so

$$\oint_C \frac{e^z}{z^4 + 5z^3}\, dz = 2\pi i \operatorname{Res}\big(f(z), 0 \big)$$

$$= 2\pi i \, \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} z^3 \frac{e^z}{z^3(z+5)}$$

$$= \pi i \lim_{z \to 0} \frac{(z^2 + 8z + 17)e^z}{(z+5)^3}$$

$$= \frac{17\pi}{125} i$$

(Example 7) Evaluation by the residue theorem

Evaluate $\oint_C \tan z\, dz$, where the contour C is the circle $|z| = 2$.

<u>*Solution*</u>*:*

The integrand $f(z) = \tan z = \dfrac{\sin z}{\cos z}$ has simple poles at the points where $\cos z = 0$. From Section 4.3, the only zeros $\cos z$ are the real numbers $z = \dfrac{(2n+1)p}{2}$, $n = 0, \pm 1, \pm 2, \cdots$. Since only $-\dfrac{p}{2}$ and $\dfrac{p}{2}$ are within the circle $|z| = 2$,

$$\oint_C \tan z\, dz = \oint_C \frac{\sin z}{\cos z} dz = \oint_C \frac{g(z)}{h(z)} dz$$

$$= 2p\, i \left[\operatorname{Res}\left(f(z), -\frac{p}{2} \right) + \operatorname{Res}\left(f(z), \frac{p}{2} \right) \right]$$

$$= 2p\, i \left[\frac{\sin\left(-\frac{p}{2}\right)}{-\sin\left(-\frac{p}{2}\right)} + \frac{\sin\left(\frac{p}{2}\right)}{-\sin\left(\frac{p}{2}\right)} \right] \quad \left(\because \operatorname{Res}\left(f(z),\ z_0 \right) = \frac{g(z_0)}{h'(z_0)} \right)$$

$$= 2p\, i\, [-1-1] = -4p\, i$$